1-Apr-2018 EQUIVALENCE RELATION

~~rediting~~ redefining "equal" i.e "=".

Eg: Let set $B = \mathbb{Z}$, defining "=" on B s.t. $a=b$ if $5 | (a-b)$ in $\mathbb{Z}$. (factors have to be Integer).

1) Show that "=" is an equivalence relation on B.
2) Find all equivalence classes of "=".
3) Does $(3,10) \in$ "=" ?
   Does $(7,12) \in$ "=" ?

Ans: 1) We need to show 3 things:

i) A—A / reflexive (txt book calls it symmetric)
   Means "$a = a$" for every $a \in B$

ii) A—B / symmetric
   Means if "$a=b$" $\forall a,b \in B$, then "$b=a$".

iii) A—B—C / transitive
   Means if "$a=b$" and "$b=c$" for some $a,b,c \in B$, then "$a=c$".

A—A: Let $a \in B$. Show "$a=a$" i.e show $5|(a-a)$.
$a-a=0$, $5|0$, $0 = 5 \times 0$, $0 \in \mathbb{Z}$.

A—B: Assume "$a=b$" for some $a,b \in B$. Show "$b=a$".
i.e assume $a-b = 5k_1$, for some $k_1 \in \mathbb{Z}$.
Multiply by -1: $b-a = 5 \times (-k_1)$, $-k_1 \in \mathbb{Z}$
Hence $b=a$.

A—B—C: Assume "$a=b$" and "$b=c$" for some $a,b,c \in \mathbb{Z}$. Show "$a=c$".
$a-b = 5k_1$ and $b-c = 5k_2$ for some $k_1, k_2 \in \mathbb{Z}$.
Add; $a-b+b-c = 5k_1 + 5k_2$
$a-c = 5(k_1+k_2)$, $k_1+k_2 \in \mathbb{Z}$.
Hence $a=c$.

$\therefore$ "=" is an equivalence relation

2) $\bar{0} = [0]$ (set of all numbers that "=" 0.)
$[0] = \{\dots, -5, 0, 5, 10, 15, \dots\}$. i.e $5n$, $n \in \mathbb{Z}$.
$[10] = [0]$
or $[100] = [0]$

$\bar{1} = [1] = \{\dots, -9, -4, 1, 6, 11, \dots\}$ set of all numbers that "=" 1.
i.e. $5n+1$, $n \in \mathbb{Z}$

$[2] = \{\dots, -8, -3, 2, 7, \cancel{13}\ 12, \dots\}$ $5n+2$, $n \in \mathbb{Z}$.

$[3] = \{\dots, -7, -2, 3, 8, 13, \dots\}$

$[4] = \{\dots, -6, -1, 4, 9, 14, \dots\}$.

Fact: intersection of any 2 distinct equivalence classes is empty.
union of all equivalence classes is whole set B.
Equivalence relation partitions the set to subsets.

contd. 5) We view elements of the new relation as a subet of $B \times B = \{(a_1, a_2) \mid a_1, a_2 \in B\}$
$(3,10) \in$ "=" means $3 = 10$.
Check: $3 - 10 = -7$
$5 \nmid -7$. Hence, $(3,10) \notin$ "=".

$(7,12) \in$ "=" means $7 = 12$
Check: $7 - 12 = -5$
$5 \mid -5$. Hence $(7,12) \in$ "=".

Homework 11:

Ques. 1: Let $A = \{0011, 1011, 0101, 0111, 1111, 1101\}$. Define = on A, where if $a, b \in A$ then $a = b$ if number of zero digits on a = no. of zero digits on b.
i) Convince me that "=" is an equivalence relation.
ii) Find all equivalence classes of $(A, "=")$
iii) View "=" as a subset of $A \times A$. How many elements does "=" have?
iv) Write down all elements of "=".

Ans: i) check:
A—A. let $a \in A$. show "$a = a$".
Meaning number of zero digits in a = number of zero digits in a. This is true by observation. $0101 = 0101$
$1011 = 1011$.
Axiom 1 holds.

A—B. let $a, b \in A$. If "$a=b$", then show "$b=a$".
Note: set A is finite. This means we can prove by example instead of by argument.
Example; let $a = 0011$ and $b = 0101$.
"$a = b$" $(0011 = 0101)$ because number of zero digits in 0011 = no. of zero digits in 0101

No. of zero digits in 0101 (b) = no. of zero digits in 0011 (a)
Hence "$b=a$". Axiom 2 holds.

A—B—C. let $a, b, c \in A$. If "$a=b$" and "$b=c$" show "$a=c$".
Example: let $a = 1011$, $b = 0111$, $c = 1101$.
$1011 = 0111$, no. of zero digits in 1011 = no. of zero digits in 0111.
$0111 = 1101$, no. of zero digits in 0111 = no. of zero digits in 1101.
$\therefore$ no. of zero digits in 1011 = no. of zero digits in 1101.
$1011 = 1101$
$a = c$. Axiom 3 holds.

Hence, "=" is an equivalence relation.

ii) $[1111] = \{1111\}$
$[1011] = \{1011, 0111, 1101\}$
$[0011] = \{0011, 0101\}$

iii) No. of elements $= 1 + 3^2 + 2^2$
$= 14$.

iv) (1111, 1111)
(1011, 1011) (1011, 0111) (1011, 1101)
(0111, 1011) (0111, 0111) (0111, 1101)
(1101, 1011) (1101, 0111) (1101, 1101)
(0011, 0011) (0011, 0101) (0101, 0011) (0101, 0101)

Ques 2: Let $A = \{1, 5, 7, 9, 16, 22\}$. Define $=$ on A, where if $a, b \in A$, then $a = b$ if $a | b$ (in A). Convince me this is not an equivalence relationship.

Ans: Check:

A–A. let $a \in A$. show "$a = a$".
let $a = 5$. $5 = 5 \times 1$, $1 \in A$. Axiom 1 holds.

A–B. let $a, b \in A$. Assume "$a = b$". Show "$b = a$".
let $a = 1$, $b = 7$. $7 = 7 \times 1$, $7 \in A$.
$1 = 7 \times \frac{1}{7}$, $\frac{1}{7} \notin A$. Axiom 2 fails to hold. ("$b \neq a$").

$\therefore$ "$=$" is not an equivalence relation on A.

Ques 3: Let $A = \{5, 7, 9, 16, 22\}$. Define "$=$" on A where if $a, b \in A$ then $a = b$ if $a | b$ (in A). Convince me this is not an equivalence relationship.

Ans: Check:

A–A. let $a \in A$. show "$a = a$".
let $a = 5$. $5 = 5 \times 1$, $1 \notin A$. Axiom 1 fails to hold.

$\therefore$ "$=$" is not an equivalence relation on A.

Ques 4: Let $A = \{5, 7, 9, 11, 19, 20\}$. Define "$=$" on A, where if $a, b \in A$, then $a = b$ if $a \pmod 4 = b \pmod 4$.

i) Convince me that "$=$" is an equivalence relation.
ii) Find all equivalence classes of $(A, =)$.
iii) View "$=$" as a subset of $A \times A$. How many elements does "$=$" have.
iv) ~~Ho~~ Write down the elements of "$=$".

Ans: i) Check:

A — A. Let $a \in A$. Show "$a = a$".
let $a = 5$. $a \bmod 4 = a \bmod 4$
i.e $5 \bmod 4 = 5 \bmod 4$
$(5 - 5) \bmod 4 = 0$
$0 \bmod 4 = 0$. Axiom 1 holds.

A–B. Let $a, b \in A$. Assume "$a = b$".
i.e $a \bmod 4 = b \bmod 4$. let $a = 5$ and $b = 9$
$(a - b) \bmod 4 = 0$
$(5 - 9) \bmod 4 = 0$
$-4 \bmod 4 = 0$
Show "$b = a$".
$\times -1$; $-(a - b) \bmod 4 = 0$
$(b - a) \bmod 4 = (9 - 5) \bmod 4 = 4 \bmod 4 = 0$.
Axiom 2 holds. ($b = a$)

A–B–C. Let $a, b, c \in A$. Assume "$a = b$" and "$b = c$".
i.e $(a - b) \bmod 4 = 0$ and $(b - c) \bmod 4 = 0$
let $a = 7$, $b = 11$, $c = 19$.
$(7 - 11) \bmod 4 = -4 \bmod 4 = 0$. $(11 - 19) \bmod 4 = -8 \bmod 4 = 0$.
Add; $(7 - 11 + 11 - 19) \bmod 4$
$= -12 \bmod 4$
$= 0$.
$\therefore$ $7 = 19$, $a = c$. Axiom 3 holds. Hence, "$=$" is an equivalence relation on A.

ii) $[5] = \{5, 9\}$
$[7] = \{7, 11, 19\}$
$[20] = \{20\}$

iii) No. of elements $= 2^2 + 3^2 + 1$
$= 14$

iv) $(5,5)$ $(5,9)$ $(9,5)$ $(9,9)$
$(7,11)$ $(7,7)$ $(7,19)$
$(11,7)$ $(11,11)$ $(11,19)$
$(19,7)$ $(19,11)$ $(19,19)$
$(20,20)$

Ques 5 Let $A = \mathbb{Z}$. Define $=$ on A, where if $a, b \in A$, then $a = b$ if $7 \mid (a-b)$ (in $\mathbb{Z}$).
i) Convince me this is an equivalence relationship.
ii) Find all equivalence classes of $(A, =)$.
iii) View $=$ as a subset of $A \times A$. Is $(3, 10) \in =$? Is $(4, 12) \in =$?

Answer: i) check:

A–A. Let $a \in A$. Show "$a = a$".
$a - a = 0 = 7 \times 0$, $0 \in \mathbb{Z}$. Axiom 1 holds.

A–B. Let $a, b \in A$. Assume "$a = b$". Show "$b = a$".
$a - b = 7 \times k_1$, for some $k_1 \in \mathbb{Z}$.
$\times -1$; $b - a = 7 \times (-k_1)$, $-k_1 \in \mathbb{Z}$.
Hence "$b = a$". Axiom 2 holds.

A–B–C. Let $a, b, c \in A$. Assume "$a = b$" and "$b = c$". Show "$a = c$".
$a - b = 7 \times k_1$, $k_1 \in \mathbb{Z}$ $\quad b - c = 7k_2$, $k_2 \in \mathbb{Z}$.
Add; $a - b + b - c = 7k_1 + 7k_2$
$a - c = 7(k_1 + k_2)$, $k_1 + k_2 \in \mathbb{Z}$.
Hence "$a = c$". Axiom 3 holds.

$\therefore$ "$=$" is an equivalence relation on A.

ii) $[0] = \{\ldots, -14, -7, 0, 7, 14, 21, \ldots\}$
$[1] = \{\ldots, -13, -6, 1, 8, 15, 22, \ldots\}$
$[2] = \{\ldots, -12, -5, 2, 9, 16, 23, \ldots\}$
$[3] = \{\ldots, -11, -4, 3, 10, 17, 24, \ldots\}$
$[4] = \{\ldots, -10, -3, 4, 11, 18, 25, \ldots\}$
$[5] = \{\ldots, -9, -2, 5, 12, 19, 26, \ldots\}$
$[6] = \{\ldots, -8, -1, 6, 13, 20, 27, \ldots\}$

iii) $3 - 10 = -7 = 7 \times -1$, $-1 \in \mathbb{Z}$
$\therefore (3, 10) \in =$.

$4 - 12 = -8 = 7 \times -\frac{8}{7}$, $-\frac{8}{7} \notin \mathbb{Z}$

$\therefore (4, 12) \notin =$.

Question 6: Let $A = \{-1, 0, 1, 7, 10, 16, 19\}$. Define "$=$" on A, where if $a, b \in A$, then $a = b$ if $3 \mid (a-b)$ (in A).

i) Convince me $=$ is an equivalence relation on A.

ii) Find all equivalence classes of $(A, =)$.

iii) View "$=$" as a subset of $A \times A$. How many elements does $=$ have?

iv) Write down all elements of $=$.

Answer: i) Note: This is a finite set. Prove by example.

Check:

A–A. This axiom holds because for every element a in A, $a - a = 3 \times 0$, and $0 \in A$.

A–B. $\forall a, b \in A$, if $a = b$, show $b = a$. Let $a = 7$, $b = 10$,
$7 - 10 = -3 = 3 \times -1$, $-1 \in A$.
$\times -1$; $10 - 7 = 3 = 3 \times 1$, $1 \in A$.
$\therefore$ "$b = a$". Axiom 2 holds. $16 = 19$ is true as well.

A–B–C. $\forall a, b, c \in A$, if $a = b$ and $b = c$, show $a = c$.
There are no elements in A for the first statement to be true. So by default, the second statement, $a = c$ is true. Axiom 3 holds.

$\therefore =$ is an equivalence relation on A.

ii) $[-1] = \{-1\}$
$[0] = \{0\}$
$[1] = \{1\}$
$[7] = \{7, 10\}$
$[16] = \{16, 19\}$

iii) No. of elements $= 1 + 1 + 1 + 2^2 + 2^2$
$= 11$

iv) $(-1,-1)$ $(0,0)$ $(1,1)$
$(7,10)$ $(7,7)$ $(10,7)$ $(10,10)$
$(16,16)$ $(16,19)$ $(19,16)$ $(19,19)$

Question 7: Let $A = \{-1, 0, 1, 7, 10, 16, 19, 22\}$. Define "$=$" on A, where if $a, b \in A$ then $a = b$ if $3 \mid (a-b)$ (in A). Convince me this is not an equivalence relationship

Answer: check.

Axioms 1 & 2 hold.

A–B–C. If $a = b$ and $b = c$ for some $a, b, c \in A$. Show $a = c$.
Let $a = 16$, $b = 19$, $c = 22$.
$16 - 19 = -3 = 3 \times -1$, $-1 \in A$.
$19 - 22 = -3 = 3 \times -1$, $-1 \in A$
Add; $16 - 19 + 19 - 22 = 3(-1) + 3(-1)$.
$16 - 22 = 3 \times (-2)$, $-2 \notin A$.
Axiom 3 does not hold. "$a \neq c$".
$\therefore =$ is not an equivalence relationship.